Some identities of higher-order Euler polynomials arising from Euler basis

by
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Abstract

The purpose of this paper is to present a systematic study of some families of higher-order Euler numbers and polynomials. In particular, by using the basis property of higher-order Euler polynomials for the space of polynomials of degree less than and equal to n, we derive some interesting identities for the higher-order Euler polynomials.

1 Introduction

As is well known, the n-th Euler polynomials of order r are defined by the generating function to be

$$(\frac{2}{e^t+1})^r e^{xt} = e^{E^{(r)}(x)t} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} \quad (r \in \mathbf{Z}_+)\,, \tag{1}$$

with the usual convention about replacing $(E^{(r)}(x))^n$ by $E_n^{(r)}(x)$ (see [1-11]). In the special case, $x=0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the n-th Euler numbers of order r.
By(1), we easily get

$$\begin{aligned} E_n^{(r)}(x) &= \sum_{l=0}^{n} \binom{n}{l} E_l^{(r)} x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}^{(r)} x^l \\ &= \sum_{n=n_1+\cdots+n_r+n_{r+1}} \binom{n}{n_1,\cdots,n_r,n_{r+1}} E_{n_1}E_{n_2}\cdots E_{n_r}x^{n_{r+1}}\,. \end{aligned} \tag{2}$$

From(2), we note that the leading coefficient of $E_n^{(r)}(x)$ is given by

$$\sum_{n_1+\cdots+n_r=0} \binom{n}{n_1, \cdots, n_r} E_{n_1} E_{n_2} \cdots E_{n_r} = 1 \,. \tag{3}$$

Thus, $E_n^{(r)}(x)$ is a monic polynomial of degree n with rational coefficients.
From(1), we have $E_n^{(0)}(x) = x^n$. It is not difficult to show that

$$\frac{dE_n^{(r)}(x)}{dx} = nE_{n-1}^{(r)}(x)\,, \quad E_n^{(r)}(x+1) + E_n^{(r)}(x) = 2E_n^{(r-1)}(x), \quad (\text{see}\,[11\text{-}18])\,. \tag{4}$$

Now, we define two linear operators $\tilde{\triangle}$ and D on the space of real-valued differentiable functions as follows:

$$\tilde{\triangle} f(x) = f(x+1) + f(x), \quad Df(x) = \frac{df(x)}{dx}\,. \tag{5}$$

Then we see that $\tilde{\triangle} D = D \tilde{\triangle}$.

Let $V_n = \{p\,(x) \in \mathbf{Q}[x] |\ \deg p\,(x) \leq n\}$ be the $(n+1)$-dimensional vector space over $\mathbf{Q}$. Probably, $\{1, x, \cdots, x^n\}$ is the most natural basis for V_n. But $\{E_0^{(r)}, E_1^{(r)}, \cdots, E_n^{(r)}\}$ is also a good basis for the space V_n for our purpose of arithmetical and combinatorial applications of the higher-order Euler polynomials.
If $p\,(x) \in V_n$, then $p(x)$ can be expressed by

$$p\,(x) = b_0 E_0^{(r)}(x) + b_1 E_1^{(r)}(x) + \cdots + b_n E_n^{(r)}(x)\,.$$

In this paper, we develop methods for computing b_l from the information of $p\,(x)$ and apply those results to arithmetically and combinatorially interesting identities involving $E_0^{(r)}, E_1^{(r)}, \cdots, E_n^{(r)}$.

2 Higher-order Euler polynomials

From(5), we have

$$\tilde{\triangle} E_n^{(r)}(x) = E_n^{(r)}(x+1) + E_n^{(r)}(x) = 2E_n^{(r-1)}(x)\,, \tag{6}$$

and

$$DE_n^{(r)}(x) = nE_{n-1}^{(r)}(x)\,. \tag{7}$$

Let us assume that $p(x) \in V_n$. Then $p(x)$ can be generated by $E_0^{(r)}(x)$, $E_1^{(r)}(x)$, $\cdots, E_n^{(r)}(x)$ to be

$$p(x) = \sum_{k=0}^{n} b_k E_k^{(r)}(x). \tag{8}$$

Thus, by(8), we get

$$\tilde{\triangle} p(x) = \sum_{k=0}^{n} b_k \tilde{\triangle} E_k^{(r)}(x) = 2 \sum_{k=0}^{n} b_k E_k^{(r-1)}(x),$$

and

$$\tilde{\triangle}^2 p(x) = 2 \sum_{k=0}^{n} b_k \tilde{\triangle} E_k^{(r-1)}(x) = 2^2 \sum_{k=0}^{n} b_k E_k^{(r-2)}(x).$$

Continuing this process, we have

$$\tilde{\triangle}^r p(x) = 2^r \sum_{k=0}^{n} b_k E_k^{(0)}(x) = 2^r \sum_{k=0}^{n} b_k x^k. \tag{9}$$

Let us take the operator D^k on (9). Then

$$\begin{aligned} D^k \tilde{\triangle}^r p(x) &= 2^r \sum_{l=k}^{n} b_l l(l-1)\cdots(l-k+1)x^{l-k} \qquad (10) \\ &= 2^r \sum_{l=k}^{n} b_l \frac{l!}{(l-k)!} x^{l-k} \\ &= 2^r \sum_{l=k}^{n} b_l k! \left(\begin{array}{c} l \\ k \end{array} \right) x^{l-k}. \end{aligned}$$

Let us take $x = 0$ on (10). Then we get

$$D^k \tilde{\triangle}^r p(0) = 2^r b_k k!. \tag{11}$$

From (11), we have

$$\begin{aligned} b_k &= \frac{1}{2^r k!} D^k \tilde{\triangle}^r p(0) = \frac{1}{2^r k!} \tilde{\triangle}^r D^k p(0) \qquad (12) \\ &= \frac{1}{2^r k!} \sum_{j=0}^{r} \left(\begin{array}{c} r \\ j \end{array} \right) D^k p(j). \end{aligned}$$

Therefore, by (8)and(12), we obtain the following theorem.

Theorem 1. *For* $n, r \in \mathbf{Z}_+$, $\quad p(x) \in V_n$, *we have*

$$p\left(x\right)=\frac{1}{2^r}\sum_{k=0}^{n}\left(\sum_{j=0}^{r}\frac{1}{k!}\left(\begin{array}{c} r \\ j \end{array}\right)D^k p\left(j\right)\right)E_k^{(r)}(x)\,.$$

Let us take $p\left(x\right)=x^n \in V_n$. Then we easily see that $D^k x^n=\frac{n!}{(n-k)!}x^{n-k}$. Thus, by Theorem 1, we get

$$\begin{aligned} x^n &= \frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\frac{1}{k!}\left(\begin{array}{c} r \\ j \end{array}\right)\frac{n!}{(n-k)!}j^{n-k}E_k^{(r)}(x) \\ &= \frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)\left(\begin{array}{c} n \\ k \end{array}\right)j^{n-k}E_k^{(r)}(x)\,. \end{aligned} \tag{13}$$

Therefore, by (13), we obtain the following corollary.

Corollary 2. *For* $n, r \in \mathbf{Z}_+$, *we have*

$$x^n=\frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)\left(\begin{array}{c} n \\ k \end{array}\right)j^{n-k}E_k^{(r)}(x)\,.$$

Let $p\left(x\right)=B_n^{(s)}(x)(s \in \mathbf{Z}_+)$. Then we have

$$D^k B_n^{(s)}(x)=\frac{n!}{(n-k)!}B_{n-k}^{(s)}(x)\,. \tag{14}$$

By Theorem 1, we get

$$B_n^{(s)}(x)=\frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)\left(\begin{array}{c} n \\ k \end{array}\right)B_{n-k}^{(s)}(j)E_k^{(r)}(x)\,. \tag{15}$$

Therefore, by(15), we obtain the following corollary.

Corollary 3. *For* $n, s, r \in \mathbf{Z}_+$, $\quad$ *we have*

$$B_n^{(s)}(x)=\frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\left(\begin{array}{c} r \\ j \end{array}\right)\left(\begin{array}{c} n \\ k \end{array}\right)B_{n-k}^{(s)}(j)E_k^{(r)}(x)\,,$$

where $B_n^{(s)}(x)$ *are the* n*-th Bernoulli polynomials of order* s.

It is well known that

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\,,\quad \frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{x^n}{n!}\,. \tag{16}$$

In the special case, $x=0$, let $B_n(0)=B_n$, $E_n(0)=E_n$. From(16), we easily derive the following identity:

$$B_n(x)=\sum_{k=0,k\neq 1}^{n}\binom{n}{k}B_kE_{n-k}(x)\in V_n\,. \tag{17}$$

Let us take $p\,(x)=B_n(x)$. Then we have

$$D^kB_n(x)=n(n-1)\cdots(n-k+1)B_{n-k}(x)=\frac{n!}{(n-k)!}B_{n-k}(x)\,. \tag{18}$$

Therefore, by Theorem 1, (17) and (18), we obtain the following theorem.

Theorem 4. *For $n,r\in\mathbf{Z}_+$, we have*

$$\sum_{k=0,k\neq 1}^{n}\binom{n}{k}B_kE_{n-k}(x)=\frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\binom{r}{j}\binom{n}{k}B_{n-k}(j)E_k^{(r)}(x)\,.$$

Let us consider $p\,(x)=\sum_{k=0}^{n}B_k(x)B_{n-k}(x)\,.$
Then we have

$$D^kp\,(x)=\frac{(n+1)!}{(n-k+1)!}\sum_{l=k}^{n}B_{l-k}(x)B_{n-l}(x)\,. \tag{19}$$

Thus, by Theorem 1 and (19), we obtain the following theorem.

Theorem 5. *For $r,n\in\mathbf{Z}_+$, we have*

$$\sum_{k=0}^{n}B_k(x)B_{n-k}(x)=\frac{1}{2^r}\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{j=0}^{r}\binom{r}{j}\binom{n+1}{k}B_{l-k}(j)B_{n-l}(j)E_k^{(r)}(x)\,.$$

Let $n,\, m \in \mathbf{Z}_+$, with $n \geq m+2$. Then we have

$$B_m(x)B_{n-m}(x)$$
$$= \sum_{l=0}^{\infty} \{ \binom{m}{2l}(n-m) + \binom{n-m}{2l} m \} \frac{B_{2l}B_{n-2l}(x)}{n-2l} + (-1)^{m+1} \frac{B_n}{\binom{n}{m}} . \tag{20}$$

Let us take $p\,(x) = B_m(x)B_{n-m}(x) \in V_n$.
Then we have

$$D^k p\,(x)$$
$$= \sum_{l=k}^{\infty} \left\{ \binom{m}{2l}(n-m) + \binom{n-m}{2l} m \right\} \frac{B_{2l}}{n–2l} \times \frac{(n-2l)!}{(n-2l-k)!} B_{n-2l-k}(x). \tag{21}$$

Therefore, by Theorem 1 and (21), we obtain the following theorem.

Theorem 6. *For $n, m \in \mathbf{Z}_+$ with $n \geq m+2$, we have*

$$B_m(x)B_{n-m}(x)$$
$$= \frac{1}{2^r} \sum_{k=0}^{n} \left\{ \sum_{l=k}^{\infty} \sum_{j=0}^{r} \binom{r}{j} \binom{n-2l}{k} \right.$$
$$\left. \times \left(\binom{m}{2l}(n-m) + \binom{n-m}{2l} m \right) \frac{B_{2l}B_{n-2l-k}(j)}{n-2l} \right\} E_k^{(r)}(x).$$

Remark. By using Theorem 1, we can find many interesting identities related to Bernoulli and Euler polynomials.

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